Math 245C Lecture 28 Notes

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1 Distributions of Differences

1.1 Differences of functions in Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. If $A \subseteq \mathbb{R}^d$, $f : A \to \mathbb{R}$¡ and $y \in \mathbb{R}^d$¡ we set $f_y(x) = f(x-y)$ for $x \in A + y$. If $\phi \in C_c^\infty(\Omega)$, let $O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\}$.

Proposition 1.1. *Let $\phi \in C_c^\infty(\Omega)$ and $y \in \mathbb{R}^d$. Then $K = \bigcup_{y \in [0,1]}(ty + \operatorname{supp}(\phi))$ is compact.*

Proof. Set $f(t,z) = ty + z$. $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is continuous, and $K = f([0,1] \times \operatorname{supp}(\phi))$ is compact as the image of a compact set by a continuous function. $\square$

Theorem 1.1. *Let $T \in \mathcal{D}'(\Omega)$, and let $y \in \mathbb{R}^d$.*

1. *If $\phi \in C_c^\infty(\Omega)$ and $ty + \operatorname{supp}(\phi) \subseteq \Omega$ for all $t \in [0,1]$, then*

$$T(\phi_y) = T(\phi) = \int_0^1 \sum_{j=1}^d y_j \partial_j T(\phi_{ty})\, dt.$$

2. *If $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, then for a.e. $x \in \mathbb{R}^d$,*

$$f(x+y) - f(x) = \int_0^1 \nabla f(x+ty) \cdot y\, dt.$$

In the second case, if we could show that $\frac{d}{dt}T(\phi_{ty}) = \nabla T(\phi_{ty}) \cdot y$ and that this derivative is continuous, we could just use the fundamental theorem of calculus.

Proof. Set $K = \bigcup_{t \in [0,1]}(ty + \operatorname{supp}(\phi))$. Then $K \subseteq \Omega$ is compact. For $x \in \mathbb{R}^d$ and $h \neq 0$,

$$L_h(x) := \frac{\phi(x - (t+h)y) - \phi(x - ty)}{h} = -\int_0^1 \nabla\phi(x - ty - \tau h y) \cdot y\, d\tau.$$

Note that $L_h \in C_c^\infty(\Omega)$ if $0 < |h| \ll 1$, and

$$\lim_{h\to 0} H_h(x) = \nabla\phi(x - ty)\cdot y =: L_0(x).$$

Also, $(L_h)_h$ converges to L_0 in $C_c^\infty(\Omega)$. Thus,

$$\begin{aligned}\frac{d}{dt}T(\phi_{ty}) &= \lim_{h\to 0} T(L_h) = T(L_0) = T(-\nabla\phi(x-ty)\cdot y)\\ &= -\sum_{j=1}^d y_j T(\partial_j\phi(\cdot - ty)) = \sum_{j=1}^d y_j \partial_j T(\phi(\cdot - ty))\\ &= \sum_{j=1}^d y_j\partial_j T(\phi_{ty}).\end{aligned}$$

As $t \to \partial_j T(\phi_{t,y})$ is continuous, we conclude that $t \to \frac{d}{dt}T(\phi_{ty})$ is continuous. So we get

$$T(\phi_y) - T(\phi) = \int_0^1 \frac{d}{dt}(T(\phi_{ty}))\,dt = \int_0^1 \nabla T(\phi_{ty})\cdot y\,dt.$$

For the second statement, let $f \in W^{1,1}_{\text{loc}}$, and set

$$T(\phi) = \int_{\mathbb{R}^d} \phi(x)f(x)\,dx.$$

Then $T \in \mathcal{D}'(\Omega)$, and $\partial_j T(\phi) = -T(\partial_j\phi) = -\int_{\mathbb{R}^d}\partial_j\phi f$. So

$$\partial_j T(\phi) = \int_{\mathbb{R}^d}\phi(x)\partial_j f(x)\,dx.$$

By the first statement,

$$\int_{\mathbb{R}^d}(\phi_y(x) - \phi(x))f(x)\,dx = \int_0^1\int_{\mathbb{R}^d}\sum_{j=1}^d y_j\phi_{ty}(x)\partial_j f(x)\,dx\,dt.$$

The left hand side is

$$\int_{\mathbb{R}^d}(\phi(x-y) - \phi(x))f(x)\,dx,$$

and the left hand side is

$$\int_0^1\int_{\mathbb{R}^d}\sum_{j=1}^d y_j\phi(x-ty)\partial_j f(x)\,dx\,dt.$$

If we make the change of variables $z = z - y$, then

$$\int_{\mathbb{R}^d} \phi(z)(f(z+y) - f(z))\,dz = \int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^{d} y_j \phi(z) \partial_j f(z+ty)\,dz\,dt.$$

Since ϕ is of compact support and $\partial_j f \in L^1_{\text{loc}}$ we check that we can apply Fubini's theorem to conclude that

$$\int_{\mathbb{R}^d} \phi(z)(f(z+y) - f(z))\,dz = \int_{\mathbb{R}^d} \phi(z) \left(\int_0^1 \nabla f(z+ty) \cdot y\,dt \right) dz.$$

By H older's inequality, this implies that $z \mapsto \int_0^1 \nabla f(z+ty) \cdot y\,dt \in L^1_{\text{loc}}(\mathbb{R}^d)$, and

$$f(z+y) - f(z) = \int_0^1 \nabla f(z+ty) \cdot y\,dt$$

for a.e. $z \in \mathbb{R}^d$. □

Remark 1.1. Let $f \in C^1(\Omega)$, and set

$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx), \qquad \phi \in C_c^\infty(\Omega).$$

Then $T \in \mathcal{D}'(\Omega)$, and

$$\partial_j T(\phi) = \int_\Omega \frac{\partial f}{\partial x_j}(x)\phi(x)\,dx,$$

where $\frac{\partial f}{\partial x_j}$ is the pointwise derivative.

This has a converse.

Theorem 1.2. *Let $g_1, \ldots, g_d \in C(\Omega)$, and let $T \in \mathcal{D}'(\Omega)$ be such that $\partial_j T = g_j$ for $j = 1, \ldots, d$. Then there exists $f \in C^1(\Omega)$ such that*

$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx, \qquad \phi \in C_c^\infty(\Omega).$$

Then

$$g_j = \frac{\partial f}{\partial x_j}.$$

Corollary 1.1. *If Ω is connected, $T \in \mathcal{D}'(\Omega)$, and $\partial_j = 0$ for $j = 1, \ldots, d$, then there exists $C \in R$ such that*

$$T(\phi) = C \int_\Omega \phi(x)\,dx, \qquad \forall \phi \in C_c^\infty(\Omega).$$